RESOLUTION OF SINGULARITIES AND GEOMETRIC PROOFS OF THE ŁOJASIEWICZ INEQUALITIES

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ABSTRACT. The Łojasiewicz inequalities for real analytic functions on Euclidean space were first proved by Stanisław Łojasiewicz in [83, 84, 87] using methods of semianalytic and subanalytic sets, arguments later simplified by Bierstone and Milman [7]. In this article, we first give an elementary geometric, coordinate-based proof of the Łojasiewicz inequalities in the special case where the function is C^1 with simple normal crossings. We then prove, partly following Bierstone and Milman [9, Section 2] and using resolution of singularities for real analytic varieties, that the gradient inequality for an arbitrary real analytic function follows from the special case where it has simple normal crossings. In addition, we give elementary proofs of the the Łojasiewicz inequalities when the function is C^2 and Morse–Bott or C^N and Morse–Bott of order $N \geq 2$.

1. INTRODUCTION

Our goal in this article is to provide geometric proofs of the Łojasiewicz inequalities (Theorem 1 and Corollaries 4 and 5) for functions with simple normal crossings and hence, via resolution of singularities, for arbitrary real analytic functions on Euclidean space. In contrast, for a function that is (generalized) Morse–Bott (so its critical set is a submanifold), elementary methods suffice to prove the Łojasiewicz inequalities (Theorems 2.2 and 2.5).

The original proofs by Stanisław Łojasiewicz of his inequalities [85, 86, 87, 88, 90] relied on the theory of semianalytic sets and subanalytic sets originated by him and further developed by Gabrièlov [38], Hardt [52, 53] and Hironaka [60, 62, 59]. The proofs due to Łojasiewicz of his inequalities are well-known to be technically difficult. The most accessible modern approaches to the inequalities were provided by Bierstone and Milman. In [7], they significantly simplify the Łojasiewicz theory of semianalytic sets and subanalytic sets and prove his gradient inequality as a consequence of technical results in that theory. In [9], they develop an approach to resolution of singularities for algebraic and analytic varieties over a field of characteristic zero that relies on blowing up and greatly simplifies the original arguments due to Hironaka et al. [3, 4, 58, 61]. They then deduce the Łojasiewicz gradient inequality as a consequence of resolution of singularities for analytic varieties and a direct verification when the critical and zero set of an analytic function is a simple normal crossing divisor.

The Łojasiewicz gradient inequality was generalized by Leon Simon [108] to a certain class of real analytic functions on a Hölder space of $C^{2,\alpha}$ sections of a finite-rank vector bundle over a

Date: This version: April 28, 2018.

2010 *Mathematics Subject Classification.* Primary 32B20, 32C05, 32C18, 32C25, 58E05; secondary 14E15, 32S45, 57R45, 58A07, 58A35.

Key words and phrases. Analytic varieties, Łojasiewicz inequalities, gradient flow, Morse–Bott functions, resolution of singularities, semianalytic sets and subanalytic sets.

The author was partially supported by National Science Foundation grant DMS-1510064 and the Simons Center for Geometry and Physics, Stony Brook, the Dublin Institute for Advanced Studies, the Institut des Hautes Études Scientifiques, Bures-sur-Yvette, and the Institute for Advanced Studies, Princeton.

closed, finite-dimensional smooth manifold. Simon's proof relied on a splitting (or Lyapunov–Schmidt reduction) of the real analytic function into a finite-dimensional part, to which the original Łojasiewicz gradient inequality could be applied, and a benign infinite-dimensional part. The resulting Łojasiewicz-Simon gradient inequality and its many generalizations and variants have played a significant role in analyzing questions such as *a*) global existence, convergence, and analysis of singularities for solutions to nonlinear evolution equations that are realizable as gradient-like systems for an energy function, *b*) uniqueness of tangent cones, and *c*) gap theorems. See Feehan [32], Feehan and Maridakis [35, 36], and Huang [65] for references and a survey of Łojasiewicz–Simon gradient inequalities for real analytic functions on Banach spaces and their many applications in applied mathematics, geometric analysis, and mathematical physics.

Our hope is that the more geometric and direct coordinate-based approaches provided in this article to proofs of the Łojasiewicz gradient inequality may yield greater insight that could be useful when endeavoring to prove gradient inequalities for functions on Banach spaces arising in geometric analysis without relying on Lyapunov–Schmidt reduction to the gradient inequality for functions on Euclidean space or attempting to extend methods specific to algebraic geometry. For example, the Łojasiewicz inequalities for the F functional on the space of hypersurfaces in Euclidean space are proved directly by Colding and Minicozzi [23, 24, 25] and by the author for the Yang–Mills energy function near the moduli space of flat connections on a principal G-bundle over a closed, smooth Riemannian manifold [34]. Applications in geometric analysis typically concern functions on infinite-dimensional manifolds and, in that context, arguments specific to semianalytic sets or subanalytic subsets or real analytic subvarieties of Euclidean space do not necessarily have analogues in infinite-dimensional geometry. Like Bierstone and Milman in [9, Section 2], we ultimately apply resolution of singularities to obtain the Łojasiewicz gradient inequality for an arbitrary real analytic function, but after directly proving the gradient inequality in simpler cases. When the function is C^N and Morse–Bott of order $N \geq 2$, we obtain a Łojasiewicz exponent $\theta = 1 - 1/N$ (see Theorems 2.2 and 2.5) and when the function is C^1 with simple normal crossings, we obtain an explicit bound for the Łojasiewicz exponent — which implies that $\theta \in [1/2, 1)$ rather than $\theta \in (0, 1)$ — together with a characterization of when θ has the optimal value 1/2.

We showed in [34, Section 4] that one can use the Mean Value Theorem to prove the Łojasiewicz gradient inequality for a C^2 Morse–Bott function on a Banach space in a context of wide applicability [34, Theorem 3]. The facts that a Morse–Bott function has a critical set which is a smooth submanifold and a Hessian which is non-degenerate on the normal bundle ensure that the Mean Value Theorem easily yields the Łojasiewicz gradient inequality (with optimal Łojasiewicz exponent 1/2). In Section 3, we prove that the Łojasiewicz gradient inequality (Theorem 3) holds for a C^1 function has simple normal crossings in the sense of Definition 1.1. We then appeal to resolution of singularities (Theorem 4.5) to show that the Łojasiewicz gradient inequality for an arbitrary real analytic function, Theorem 1, is a straightforward consequence of Theorem 3. This incremental approach makes it clear that the essential difficulty is due neither to the high dimension of the ambient Euclidean space nor the critical set, but instead due (as should be expected) to possibly complicated singularities in the critical set.

Simplifications of Łojasiewicz's proofs [87] of his inequalities have also been given by Kurdyka and Parusiński [75], where they use the fact that a subanalytic set in Euclidean space admits a strict Thom stratification. Łojasiewicz and Zurro [91] further simplified the arguments of Kurdyka and Parusiński to prove the Łojasiewicz inequalities, again using properties of subanalytic sets.

The problem of estimating Łojasiewicz exponents or determining their properties, often for restricted classes of functions (for example, polynomials, certain analytic functions, functions with isolated critical points, and so on), has been pursued by many researchers, including Abderrahmane [1], Bivià-Ausina [11], Bivià-Ausina and Encinas [12, 13, 14], Bivià-Ausina and Fukui [15], Brzostowski [18], Brzostowski, Krasiński, and Oleksik [19], Búi and Pham [20], D'Acunto and Kurdyka [28], Fukui [37], Gabrièlov [39] Gwoździewicz [45], Haraux [46, Theorem 3.1], Haraux and Pham [50, 51], Ji, Kollár, and Shiffman [66] Kollár [69], Krasiński, Oleksik, and Płoski [71], Kuo [73], Lichtin [81], Lenarcik [78, 79], Lion [82], Oka [94], Oleksik [95], Pham [97, 98], Płoski [99], Risler and Trotman [100], Rodak and Spodzieja [101], [111], Tan, Yau, and Zuo [112], Teissier [113], and articles cited therein. Recently, simpler coordinate-based proofs of more limited versions of resolution of singularities for zero sets of real analytic functions, with applications to analysis, have been given by Collins, Greenleaf, and Pramanik [26] and Greenblatt [42]. In particular, Greenblatt [42, p. 1959] applies his version of resolution of singularities to prove the Łojasiewicz inequality (1.2) for a pair of real analytic functions where the zero set of one is contained in the zero set of the other. Bivià-Ausina and Encinas [12] use a resolution of singularities algorithm to estimate Łojasiewicz exponents.

Łojasiewicz [83, 84] applied his distance inequality (Corollary 4) to prove the Division Conjecture of Schwartz [104, p. 181], [105, p. 116]. In [85], he used his gradient inequality (Theorem 1) to give a positive answer to a question of Whitney: If $\mathscr{E}$ is a real analytic function on an open set $U \subset \mathbb{R}^d$, then $\mathscr{E}^{-1}(0)$ is a deformation retract of its neighborhood. This deformation retract is obtained using the negative gradient flow defined by $\mathscr{E}$. He also applies his inequalities to show that every (locally closed) semianalytic set in Euclidean space admits a Whitney stratification[1] [87, Proposition 3, p. 97 (71)]. The Łojasiewicz gradient inequality (Theorem 1) was used by Kurdyka, Mostowski, and Parusiński [74] to prove the Gradient Conjecture of Thom.

1.1. **Main results.** We now state the main results to be proved in this article, categorized according to whether or not their proofs appeal to resolution of singularities.

1.1.1. *Gradient inequality using resolution of singularities.* We begin with the fundamental

Theorem 1 (Łojasiewicz gradient inequality for an analytic function). *(See Łojasiewicz* [87, Proposition 1, p. 92 (67)]*.) Let $d \geq 1$ be an integer, $U \subset \mathbb{R}^d$ be an open subset, and $\mathscr{E} : U \to \mathbb{R}$ be a real analytic function. If $x_\infty \in U$ is a point such that $\mathscr{E}'(x_\infty) = 0$, then there are constants $C_0 \in (0, \infty)$, and $\sigma_0 \in (0, 1]$, and $\theta \in [1/2, 1)$ such that the differential map, $\mathscr{E}' : U \to \mathbb{R}^{d*}$, obeys*

$$\|\mathscr{E}'(x)\|_{\mathbb{R}^{d*}} \geq C_0 |\mathscr{E}(x) - \mathscr{E}(x_\infty)|^\theta, \quad \forall\, x \in B_{\sigma_0}(x_\infty), \tag{1.1}$$

where $\mathbb{R}^{d*} = (\mathbb{R}^d)^*$, the dual space of $\mathbb{R}^d$ and $B_{\sigma_0}(x_\infty) := \{x \in \mathbb{R}^d : \|x - x_\infty\|_{\mathbb{R}^d} < \sigma_0\} \subset U$.

Theorem 1 was stated by Łojasiewicz in [85, Theorem 4] and proved by him as [87, Proposition 1, p. 92]; see also Łojasiewicz [90, p. 1592]. Bierstone and Milman provided simplified proofs as [7, Proposition 6.8] and [9, Theorem 2.7]. Their strategy in [7] is to first prove a Łojasiewicz inequality [7, Theorem 6.4] of the form

$$|g(x)| \geq C|f(x)|^\lambda, \quad \forall\, x \in B_\sigma, \tag{1.2}$$

where f, g are subanalytic functions on an open neighborhood $U \subset \mathbb{R}^d$ of the origin such that $g^{-1}(0) \subset f^{-1}(0)$ and $B_\sigma \subset U$ and $\lambda \in (0, \infty)$. They then deduce a Łojasiewicz gradient inequality [7, Theorem 6.8] for a real analytic function f with $f'(0) = 0$,

$$\|f'(x)\|_{\mathbb{R}^{d*}} \geq C|f(x)|^\nu, \quad \forall\, x \in B_\sigma, \tag{1.3}$$

[1]The first page number refers to the version of Łojasiewicz's original manuscript mimeographed by IHES while the page number in parentheses refers to the cited LaTeX version of his manuscript prepared by M. Coste and available on the Internet.

with $\nu \in (0,1)$ by choosing $g = \|f'\|_{\mathbb{R}^{d*}}$. In [9, Theorem 2.5], the authors establish (1.2) for a pair of analytic functions by using resolution of singularities to reduce to the case that the ideal in the ring of analytic functions, $\mathscr{O}_X$, generated by fg has simple normal crossings. In [9, Theorem 2.7], they then obtain (1.3) for an analytic function f with $f(0) = 0$ and $f'(0) = 0$ by choosing $g = \|f'\|^2_{\mathbb{R}^{d*}}$ and applying (1.2) to the pair of functions f^2 (replacing g) and f^2/g (replacing f) and proving that $f^{-1}(0) \subset (f^2/g)^{-1}(0)$ and $\nu = 1/\lambda \in (0,1)$, after employing resolution of singularities to the ideal $fg\mathscr{O}_X$.

Our more direct proof of Theorem 1 makes it clear that one always has $\theta \geq 1/2$, whereas previous proofs only give $\theta \in (0,1)$. For applications to Geometric Analysis and Topology, it is essential to have $\theta < 1$, with $\theta = 1/2$ being the optimal exponent, corresponding to exponential convergence for the negative gradient flow defined by $\mathscr{E}$. In particular, we have the

Corollary 2 (Characterization of the optimal exponent and Morse–Bott condition)**.** *Assume the hypotheses of Theorem 1. If $\theta = 1/2$ then, after possibly shrinking U, there are an open neighborhood of the origin, $\widetilde{U} \subset \mathbb{R}^d$, and a real analytic map, $\pi : \widetilde{U} \to U$, such that π is a real analytic diffeomorphism on the complement of a coordinate hyperplane or the intersection of two coordinate hyperplanes and $\pi^*\mathscr{E}$ is Morse–Bott at the origin in the sense of Definition 2.1.*

See the author's [32, Theorem 3] for the statement and proof of a very general convergence-rate result for an abstract gradient flow on a Banach space defined by an analytic function obeying a Łojasiewicz–Simon gradient inequality with exponent $\theta \in [1/2, 1)$ and for previous versions of related convergence-rate results, see Chill, Haraux, and Jendoubi [22, Theorem 2], Haraux, Jendoubi, and Kavian [49, Propositions 3.1 and 3.4], Huang [65, Theorem 3.4.8], and Råde[2] [103, Proposition 7.4]. Convergence-rate results related to [32, Theorem 3] are implicit in Adams and Simon [2] and Simon [108, 109, 110], although we cannot find an explicit statement like this in those references.

1.1.2. *Gradient inequality without using resolution of singularities.* The proof of Theorem 1 in full generality provided in this article employs embedded resolution of singularities (partly following Bierstone and Milman [9, Section 2]), but there are several weaker gradient inequalities that can be proved by far more elementary methods and those provide insight to applications in geometric analysis. We now describe several results of this kind. For example, when the function $\mathscr{E}$ in Theorem 1 is C^2 (respectively, C^N with $N \geq 2$) and Morse–Bott (respectively, Morse–Bott of order N), rather than an arbitrary real analytic function, one obtains the Łojasiewicz gradient inequality with exponent $\theta = 1/2$ (respectively, $\theta = 1 - 1/N$) as a consequence of the Mean Value Theorem (respectively, Taylor Theorem): see Theorems 2.2 and 2.5. We refer the reader to Section 2 for a discussion of the Morse–Bott condition and some its generalizations, together with the statements and proof of Theorems 2.2 and 2.5.

A first reading of the proof of Theorem 2.5, which is based on a direct application of the Taylor Theorem, might suggest that it would extend to the case where $\mathscr{E}$ is an analytic function and $U \cap \operatorname{Crit}\mathscr{E}$ is an arbitrary real analytic subvariety. However, one finds that this is a more difficult strategy to develop than one might naively expect. Instead, as a stepping stone towards Theorem 1, we shall first establish a special case that holds for a class of C^1 functions. By analogy with Collins, Greenleaf, and Pramanik [26, Definition 2.5], we make the

Definition 1.1 (Function with simple normal crossings)**.** A function $f : U \to \mathbb{R}$ on an open neighborhood of the origin in $\mathbb{R}^d$ has *simple normal crossings* if

$$f(x) = x_1^{n_1} \cdots x_d^{n_d} f_0(x), \quad \forall\, x = (x_1, \dots, x_d) \in \mathbb{R}^d, \tag{1.4}$$

[2]A former postdoctoral assistant professor at Stanford University who was mentored by Simon.

where $n_i \in \mathbb{Z} \cap [0, \infty)$ and f_0 is a function such that $f_0(0) \neq 0$.

See Sections 4.1 and 4.2 for a review of normal crossings and simple normal crossings divisors in (real or complex) analytic geometry.

Theorem 3 (Łojasiewicz gradient inequality for a C^1 function with simple normal crossings and characterization of the optimal exponent and Morse–Bott condition)**.** *Let $d \geq 2$ be an integer, $U \subset \mathbb{R}^d$ be an open neighborhood of the origin, and $\mathscr{E} : U \to \mathbb{R}$ be a C^1 function with simple normal crossings. If $\mathscr{E}'(0) = 0$, then there are constants $C_0 \in (0, \infty)$ and $\sigma \in (0, 1]$ such that*

$$\|\mathscr{E}'(x)\|_{\mathbb{R}^{d*}} \geq C_0 |\mathscr{E}(x)|^{\theta}, \quad \forall\, x \in B_\sigma, \tag{1.5}$$

where $\theta = 1 - 1/N \in [1/2, 1)$ and $N = \sum_{i=1}^{c} n_i$ is the total degree of the monomial in the expression (1.4) *for $\mathscr{E}$, with $c \leq d$ being the number of exponents $n_i \geq 1$ and obeying $c \geq 2$ or $c = 1$ and $n_1 \geq 2$. In particular, $\theta = 1/2$ if and only if $c = 2$ and (after relabeling coordinates), $n_1 = n_2 = 1$ or $c_1 = 1$ and $n_1 = 2$, and furthermore, if $\mathscr{E}$ is C^2, then it is Morse–Bott on B_σ.*

Remark 1.2 (Geometry of the critical set). Theorem 2.2 shows that, when $\mathscr{E}$ is Morse–Bott and so its critical set is a smooth submanifold, then its Łojasiewicz exponent θ is equal to $1/2$. Conversely, when $\theta = 1/2$, Theorem 3 implies that $B_\sigma \cap \operatorname{Crit} \mathscr{E} = \{x_1 = 0\} \cap B_\sigma$ or $\{x_1 = x_2 = 0\} \cap B_\sigma$, a codimension one or two smooth submanifold of B_σ. Theorem 1 is proved by applying resolution of singularities to an ideal defined by an arbitrary real analytic function $\mathscr{E}$ and applying Theorem 3 to the resulting monomial (the product of $x_1^{n_1} \cdots x_d^{n_c}$ and a non-vanishing analytic function). Consequently, if $\theta = 1/2$ then there is a constraint on the nature of the singularities in the critical set of $\mathscr{E}$. Our proof of Theorem 1 shows that application of resolution of singularities does not change the Łojasiewicz exponent and so it would be of interest to try to characterize the class of real analytic functions with $\theta = 1/2$. As noted in our Introduction, the problem of computing or estimating Łojasiewicz exponents remains a topic of active research.

Our proof of Theorem 3 is a direct coordinate-based alternative to an argument due to Bierstone and Milman [9, Section 2] and relies only on the Generalized Young Inequality (3.7) (see Remark (3.1)). We are grateful to Alain Haraux for pointing out that the value for θ in previous versions of this article could be improved to the value now stated in Theorem 3 and for alerting us to his [46, Theorem 3.1]. His result is more closely related to Theorem 3 than we had realized (it assumes $f_0 = 1$ in the expression (1.4)) and we were unaware that his proof also uses the Generalized Young Inequality.

1.1.3. *Consequences of the gradient inequality.* Regardless of how proved, the gradient inequality (1.1) easily yields two useful corollaries. Note that if $\mathscr{E}(x)$ is differentiable at $x = x_0$ and $\mathscr{E}(x_0) = 0$, then $\mathscr{E}(x)^2$ has a critical point $x = x_0$.

Corollary 4 (Łojasiewicz distance inequality)**.** *(See* [87, Theorem 2, p. 85 (62)]*.) Let $d \geq 1$ be an integer, $U \subset \mathbb{R}^d$ be an open subset, and $\mathscr{E} : U \to \mathbb{R}$ be a C^∞ function. If $x_0 \in Z := \mathscr{E}^{-1}(0)$ and the gradient inequality* (1.1) *holds with exponent θ for the function $\mathscr{E}^2$ near x_0, then there are constants $C_1 \in (0, \infty)$, and $\sigma_1 \in (0, 1]$, and $\alpha = 1/(2(1 - \theta)) \in [1, \infty)$ such that*

$$|\mathscr{E}(x)| \geq C_1 \operatorname{dist}(x, Z)^{\alpha}, \quad \forall\, x \in B_{\sigma_1}(x_0), \tag{1.6}$$

where $\operatorname{dist}(x, Z) := \inf\{\|x - z\|_{\mathbb{R}^d} : z \in Z\}$*. If $\mathscr{E}'(x_0) = 0$ and the gradient inequality* (1.1) *holds with exponent θ for the function $\mathscr{E}$ near x_0, then $\alpha = 1/(1 - \theta) \in [2, \infty)$.*

Corollary 4 was stated by Łojasiewicz in [85, Corollary, p. 88] and proved by him in [83, 84] and in [87], with simplified proofs provided by Bierstone and Milman as [7, Theorem 6.4 and Remark

6.5] and [9, Theorem 2.8]. When $\mathscr{E}$ is a polynomial on $\mathbb{R}^d$, Corollary 4 is due to Hörmander [64, Lemma 1]. The following corollary is obtained by combining Theorem 1 and Corollary 4.

Corollary 5 (Łojasiewicz gradient-distance inequality)**.** *Let $d \geq 1$ be an integer, $U \subset \mathbb{R}^d$ be an open neighborhood of a point x_∞, and $\mathscr{E} : U \to \mathbb{R}$ be a C^∞ function. If $\mathscr{E}'(x_\infty) = 0$ and the gradient inequality* (1.1) *holds for $\mathscr{E}$ near x_∞, then there are constants $C_2 \in (0, \infty)$, and $\sigma_2 \in (0, 1]$, and $\beta \in [1, \infty)$ such that the differential map, $\mathscr{E}' : U \to \mathbb{R}^{d*}$, obeys*

$$\|\mathscr{E}'(x)\|_{\mathbb{R}^{d*}} \geq C_2 \operatorname{dist}(x, \operatorname{Crit} \mathscr{E})^\beta, \quad \forall\, x \in B_{\sigma_2}(x_\infty). \tag{1.7}$$

The inequality (1.7) is stated by Simon in [108, Equation (2.3)] and attributed by him to Łojasiewicz [87].

1.1.4. *Counterexamples.* It is known but worth remembering that the Łojasiewicz gradient inequality fails in general for functions that are smooth but not real analytic. For example, De Lellis [29] notes that when $d = 1$, then the function

$$\mathscr{E}(x) = \begin{cases} e^{1/|x|}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

is C^∞ on $\mathbb{R}$ with $\operatorname{Crit} \mathscr{E} = \{0\}$ but that inequalities (1.1) and (1.6) fail on any neighborhood of the origin. When $d = 2$, Haraux shows in [46, Proposition 5.2] that for the C^1 function,

$$\mathscr{E}(x, y) = \begin{cases} (x^2 + y^2)e^{-(x^2+y^2)/x^2}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

the inequality (1.1) fails on any neighborhood of the origin. Moreover, failure of a smooth function to satisfy the Łojasiewicz gradient inequality may result in non-convergence of its negative gradient flow: see Haraux [46, Remark 5.5] (citing Palis and de Melo [96]), Haraux and Jendoubi [48, Section 12.8], and Lerman [80] (citing [96, p. 14]).

1.2. **Outline.** We begin in Section 2 with elementary proofs of the Łojasiewicz gradient inequality for functions that are Morse–Bott (Theorem 2.2) or generalized Morse–Bott (Theorem 2.5). In Section 3, we establish the Łojasiewicz gradient inequality (Theorem 3) for C^1 functions with simple normal crossings. In Section 4, we review the resolution of singularities for real analytic varieties (Theorem 4.5) and apply that and Theorem 3 to prove the Łojasiewicz gradient inequality for an arbitrary real analytic function (Theorem 1). Finally, in Section 5 we deduce Corollaries 4 and 5 from the gradient inequality (1.1).

1.3. **Acknowledgments.** I thank Kenji Fukaya for his description of an example that alerted me to an error in an entirely different approach described in an early draft of this article. I am grateful to Lev Borisov, Tristan Collins, Antonella Grassi, Michael Greenblatt, Claude LeBrun, Johan de Jong, Tom Mrowka, Graeme Wilkin, and Jarek Włodarczyk for helpful communications or discussions, to Carles Bivià-Ausina, Santiago Encinas, Alain Haraux and Dennis Sullivan for many helpful comments and suggestions, to Manousos Maridakis for engaging conversations regarding Łojasiewicz inequalities, to Toby Colding and Mark Goresky for their interest in this work, and to our librarian at Rutgers, Mei-Ling Lo, for locating difficult to find articles on my behalf. I thank the anonymous referees for their careful reading of our manuscript and for their suggestions. I am grateful to the National Science Foundation for their support and to the Simons Center for Geometry and Physics, Stony Brook, the Dublin Institute for Advanced Studies, the Institut des Hautes Études Scientifiques, Bures-sur-Yvette, and the Institute for Advanced Studies, Princeton, for their hospitality and support during the preparation of this article.

2. Łojasiewicz gradient inequalities for generalized Morse–Bott functions

In this section, we adapt our previous proof in [34] of the Łojasiewicz inequalities for Morse–Bott functions on Banach spaces [34, Theorem 3] to prove the Łojasiewicz gradient inequality for Morse–Bott and generalized Morse–Bott functions, namely Theorems 2.2 and Theorem 2.5; our [34, Theorem 3] improves upon [35, Theorems 3 and 4] and has a simpler proof. Theorem 2.2 was proved by Simon [110, Lemma 3.13.1] (for a harmonic map energy function on a Banach space of $C^{2,\alpha}$ sections of a Riemannian vector bundle), Haraux and Jendoubi [47, Theorem 2.1] (for functions on abstract Hilbert spaces) and in more generality by Chill in [21, Corollary 3.12] (for functions on abstract Banach spaces). These authors do not use Morse–Bott terminology but their hypotheses imply this condition — directly in the case of Haraux and Jendoubi and Chill and by a remark due to Simon in [110, p. 80] that his integrability condition [110, Equation (iii), p. 79] is equivalent to a restatement of the Morse–Bott condition. See Feehan [33, Remark 1.16 and Appendix C] for further discussion of the relationship between definitions of integrability, such as those described by Adams and Simon [2], and the Morse–Bott condition. The method of proof of the Łojasiewicz gradient inequality for a (generalized) Morse–Bott function on Euclidean space that we give here is elementary, avoiding an appeal to the Morse–Bott Lemma (see, for example, Nicolaescu [93, Proposition 2.42]) and avoiding additional technical hypotheses required to establish a Łojasiewicz gradient inequality for a Morse–Bott function on an infinite-dimensional Banach space. A strictly weaker version of Theorem 2.5 for functions on Banach spaces (and thus quite technical hypotheses) is proved by Huang as [65, Theorem 2.4.3 and Proposition 2.7.1].

2.1. Morse–Bott and generalized Morse–Bott functions.

We begin with the

Definition 2.1 (Morse–Bott function)**.** Let $d \geq 1$ be an integer, $U \subset \mathbb{R}^d$ be an open subset, $\mathscr{E} : U \to \mathbb{R}$ be a C^2 function, and $\operatorname{Crit}\mathscr{E} := \{x \in U : \mathscr{E}'(x) = 0\}$. We say that $\mathscr{E}$ is *Morse–Bott at a point* $x_\infty \in \operatorname{Crit}\mathscr{E}$ *if a*) $\operatorname{Crit}\mathscr{E}$ is a C^2 submanifold of U, and *b*) $T_{x_\infty}\operatorname{Crit}\mathscr{E} = \operatorname{Ker}\mathscr{E}''(x_\infty)$.

In applications to Topology (see, for example, Austin and Braam [5, Section 3.1] for equivariant Floer cohomology and Bott [17] for the Periodicity Theorem), our local Definition 2.1 is often augmented by conditions that $\operatorname{Crit}\mathscr{E}$ be compact, as in Bott [16, Definition, p. 248], or compact and connected as in Nicolaescu [93, Definition 2.41], and that $T_x\operatorname{Crit}\mathscr{E} = \operatorname{Ker}\mathscr{E}''(x)$ for all $x \in \operatorname{Crit}\mathscr{E}$.

Theorem 2.2 (Łojasiewicz gradient inequality for a Morse–Bott function on Euclidean space)**.** *(Compare Feehan* [34, Theorem 3] *and Feehan and Maridakis* [35, Theorems 3 and 4] *for the case of a Morse-Bott function on a Banach space.) Let $d \geq 1$ be an integer and $U \subset \mathbb{R}^d$ an open subset. If $\mathscr{E} : U \to \mathbb{R}$ is a Morse–Bott function at a point $x_\infty \in \operatorname{Crit}\mathscr{E}$, then there are constants $C_0 \in (0,\infty)$ and $\sigma_0 \in (0,1]$ such that*

$$\|\mathscr{E}'(x)\|_{\mathbb{R}^{d*}} \geq C_0|\mathscr{E}(x) - \mathscr{E}(x_\infty)|^{1/2}, \quad \forall\, x \in B_{\sigma_0}(x_\infty). \tag{2.1}$$

Even when $\mathscr{E}$ is a C^2 Morse–Bott function on a *Banach* space, the proof of the corresponding Łojasiewicz gradient inequality [34, Theorem 3] still readily follows from the Mean Value Theorem (see [34, Section 4]) in the presence of a few additional technical hypotheses specific to the infinite-dimensional setting.

Remark 2.3 (On the proof of Theorem 2.2). Theorem 2.2 can be obtained as an immediate consequence of the Morse–Bott Lemma (see Banyaga and Hurtubise [6, Theorem 2] or Feehan [33] and references therein). However, the proof of the Morse–Bott Lemma itself (especially for Morse–Bott functions that are at most C^2) requires care whereas our proof of Theorems 2.2 and 2.5 is direct and elementary.

Definition 2.4 (Generalized Morse–Bott function)**.** Let $d \geq 1$ and $N \geq 2$ be integers, $U \subset \mathbb{R}^d$ be an open subset, and $\mathscr{E} : U \to \mathbb{R}$ be a C^2 function. We call $\mathscr{E}$ a *generalized Morse–Bott function of order* N *at a point* $x_\infty \in \operatorname{Crit}\mathscr{E}$ if *a*) $\operatorname{Crit}\mathscr{E}$ is a C^N submanifold of U, and *b*) $\mathscr{E}^{(N)}(x_\infty)v^N \neq 0$ and $\mathscr{E}^{(n)}(x_\infty)v^n = 0$ for all $v \in N_{x_\infty}\operatorname{Crit}\mathscr{E}$ and $2 \leq n \leq N$, where $N_{x_\infty}\operatorname{Crit}\mathscr{E} \subset \mathbb{R}^d$ is the orthogonal complement of the subspace $T_{x_\infty}\operatorname{Crit}\mathscr{E} \subset \mathbb{R}^d$.

For example, if $N \geq 2$ and $f(x,y) = x^N$ then $f : \mathbb{R}^2 \to \mathbb{R}$ is an generalized Morse–Bott function of order N. The analogous definition of a generalized Morse function is stated, for example, by Rothe [102, Definition 2.6] and Kuiper [72, p. 202, Corollary].

Theorem 2.5 (Łojasiewicz gradient inequality for a generalized Morse–Bott function on Euclidean space)**.** *Let $d \geq 1$ and $N \geq 2$ be integers and $U \subset \mathbb{R}^d$ be an open neighborhood. If $\mathscr{E} : U \to \mathbb{R}$ is a generalized Morse–Bott function of order N at a point $x_\infty \in \operatorname{Crit}\mathscr{E}$, then there are constants $C_0 \in (0,\infty)$ and $\sigma_0 \in (0,1]$ such that*

$$\|\mathscr{E}'(x)\|_{\mathbb{R}^{d*}} \geq C_0|\mathscr{E}(x) - \mathscr{E}(x_\infty)|^{1-1/N}, \quad \forall\, x \in B_{\sigma_0}(x_\infty). \tag{2.2}$$

As Definition 2.4 suggests, the proof of Theorem 2.5 should generalize to the setting of functions on Banach spaces, as in [34, Theorem 3] for the case of Morse–Bott functions. There are other extensions of the concept of a Morse–Bott function, notably that of Kirwan [68] and Holm and Karshon provide a version of her definition of a *Morse–Bott–Kirwan function* in [63, Definitions 2.1 and 2.3] and explore its properties and applications to Topology. However, it is unclear that the relatively simple proof of Theorem 2.2 would extend to include such Morse–Bott–Kirwan functions.

2.2. **Łojasiewicz gradient inequalities for Morse–Bott and generalized Morse–Bott functions.** Theorem 2.5 reduces to Theorem 2.2 when $N = 2$, so it suffices to give the

Proof of Theorem 2.5. We begin with four reductions that simplify the proof. First, observe that if $\mathscr{E}_0 : U \to \mathbb{R}$ is defined by $\mathscr{E}_0(x) := \mathscr{E}(x + x_\infty)$, then $\mathscr{E}_0'(0) = 0$, so we may assume without loss of generality that $x_\infty = 0$ and relabel $\mathscr{E}_0$ as $\mathscr{E}$.

Second, denote $K := \operatorname{Ker}\mathscr{E}''(0) \subset \mathbb{R}^d$ and observe that by applying a C^2 diffeomorphism to a neighborhood of the origin in $\mathbb{R}^d$ and possibly shrinking U, we may assume without loss of generality that $U \cap \operatorname{Crit}\mathscr{E} = U \cap K$, recalling that $K = T_{x_\infty}\operatorname{Crit}\mathscr{E}$ by hypothesis that $\mathscr{E}$ is Morse–Bott at x_∞.

Third, if $x \in U \cap K$, then $\mathscr{E}(x) = \mathscr{E}(0)$, and we may restrict our attention to $x \in U \cap K^\perp$ without loss of generality in the remainder of the proof. To see this, observe that if $x = x_\perp + x_\parallel \in U \cap (K^\perp \oplus K)$ then $\mathscr{E}(x_\parallel) = \mathscr{E}(0)$ (because $\mathscr{E}$ is constant along $U \cap K$) and

$$\|\mathscr{E}'(x)\|_{\mathbb{R}^{d*}} \geq C|\mathscr{E}(x) - \mathscr{E}(0)|^{1/2} \iff \|\mathscr{E}'(x)\|_{\mathbb{R}^{d*}} \geq C|\mathscr{E}(x) - \mathscr{E}(x_\parallel)|^{1/2}.$$

For each fixed $x_\parallel \in K$, we may define $\bar{\mathscr{E}}(x_\perp) := \mathscr{E}(x_\perp + x_\parallel)$ and thus $\mathscr{E}'(x_\perp + x_\parallel) = \bar{\mathscr{E}}'(x_\perp) + 0 \in K^{\perp,*} \oplus K^* = \mathbb{R}^{d*}$ for $x = x_\perp + x_\parallel \in U$, then

$$\|\mathscr{E}'(x)\|_{\mathbb{R}^{d*}} \geq C|\mathscr{E}(x) - \mathscr{E}(x_\parallel)|^{1/2} \iff \|\bar{\mathscr{E}}'(x_\perp)\|_{\mathbb{R}^{d*}} \geq C|\bar{\mathscr{E}}(x_\perp) - \bar{\mathscr{E}}(0)|^{1/2},$$

noting that $\|\bar{\mathscr{E}}'(x_\perp)\|_{\mathbb{R}^{d*}} = \|\bar{\mathscr{E}}'(x_\perp)\|_{K^{\perp,*}}$. Because $\mathscr{E}$ is constant along K, the constant C in the preceding inequalities will be independent of $x_\parallel$. We now relabel $\bar{\mathscr{E}}$ as $\mathscr{E}$.

Fourth, observe that if $\mathscr{E}_0 : U \to \mathbb{R}$ is defined by $\mathscr{E}_0(x) := \mathscr{E}(x) - \mathscr{E}(0)$, then $\mathscr{E}_0(0) = 0$, so we may once again relabel $\mathscr{E}_0$ as $\mathscr{E}$ and assume without loss of generality that $\mathscr{E}(0) = 0$.

By the second reduction above, it suffices to consider the cases where *a*) $U \cap \operatorname{Crit}\mathscr{E} = (\mathbb{R}^c \oplus 0) \cap U$, for $d \geq 2$ and $1 \leq c \leq d-1$, or *b*) $U \cap \operatorname{Crit}\mathscr{E} = 0 \in \mathbb{R}^d$, for $d \geq 1$ and $c = 0$. By the third reduction above, it suffices when $d \geq 2$ and $1 \leq c \leq d-1$ to consider the case where x is orthogonal to the subspace $\mathbb{R}^c \oplus 0 \subset \mathbb{R}^d$, so $x = 0 + x_\perp \in \mathbb{R}^c \oplus \mathbb{R}^{d-c}$. (If $d=1$, then $c=0$.)

Applying the Taylor Formula [76, p. 349] to a C^{M+1} function $f : U \to \mathbb{R}^k$ (for $k \geq 1$) gives

$$f(x) = f(0) + f'(0)x + \cdots + \frac{f^{(M)}(0)}{M!}x^M + \frac{1}{M!}\int_0^1 f^{M+1}(tx)x^{M+1}\,dt, \quad \forall\, x \in B_R, \tag{2.3}$$

for the open subset $U \subset \mathbb{R}^d$, open ball $B_R = \{x \in \mathbb{R}^d : \|x\|_{\mathbb{R}^d} < R\} \subset U$ for some $R > 0$, and integer $M \geq 0$. For *a*) $d \geq 2$ and $0 \leq c \leq d-2$, consider $v \in S^{d-1-c} = \{x \in \mathbb{R}^d : c = 0 \text{ or } x_i = 0 \text{ for } 1 \leq i \leq c \text{ and } x_{c+1}^2 + \cdots + x_d^2 = 1\}$, and for *b*) $d \geq 1$ and $c = d-1$, consider $v = \pm 1$. If $\mathscr{E}$ is constant in an open neighborhood of $0 \in \mathbb{R}^d$, then (2.2) obviously holds, so we may assume without loss of generality that $\mathscr{E}$ is non-constant in an open neighborhood of 0. Now $\mathscr{E}^{(N)}(0) \in \operatorname{Hom}_{\mathbb{R}}(\otimes^N \mathbb{R}^d, \mathbb{R}) = \otimes^N \mathbb{R}^{d*}$ is a continuous operator and by hypothesis that $\mathscr{E}$ is generalized Morse–Bott of order N at the origin, there is a positive constant $\zeta > 0$ such that

$$|\mathscr{E}^{(N)}(0)v^N| \geq \zeta, \quad \forall\, v \in S^{d-1-c}. \tag{2.4}$$

Note that, for any $v \in S^{d-1-c}$,

$$\|\mathscr{E}^{(N)}(0)v^{N-1}\|_{\mathbb{R}^{d*}} = \max_{w \in S^{d-1-c}} |\mathscr{E}^{(N)}(0)v^{N-1}w| \geq |\mathscr{E}^{(N)}(0)v^N|. \tag{2.5}$$

The lower bounds in (2.4) and (2.5) and continuity of the operator $\mathscr{E}^{(N)}(0) \in \operatorname{Hom}_{\mathbb{R}}(\otimes^{N-1}\mathbb{R}^d, \mathbb{R}^{d*})$ also ensure that

$$\|\mathscr{E}^{(N)}(0)v^{N-1}\|_{\mathbb{R}^{d*}} \geq \zeta, \quad \forall\, v \in S^{d-1-c}. \tag{2.6}$$

Because $\mathscr{E}^{(n)}(0)v^n = 0$ for $n = 1, \ldots, N-1$, the Taylor Formula (2.3) applied to $f(x) = \mathscr{E}(x)$ with $k=1$ and $M=N$ and $x = rv$ gives

$$\mathscr{E}(rv) = r^N \frac{\mathscr{E}^{(N)}(0)v^N}{N!} + \frac{r^{N+1}}{N!}\int_0^1 \mathscr{E}^{N+1}(trv)v^{N+1}\,dt, \quad \forall\, r \in [0, R). \tag{2.7}$$

We may choose $\rho \in (0, R \wedge 1]$ small enough[3] that

$$\rho \max_{s \in [0,R]} |\mathscr{E}^{N+1}(sv)v^{N+1}| \leq |\mathscr{E}^{(N)}(0)v^N|, \quad \forall\, v \in S^{d-1-c}, \tag{2.8}$$

namely, letting $\|\cdot\|$ denote the operator norm on $\operatorname{Hom}_{\mathbb{R}}(\otimes^{N+1}\mathbb{R}^d, \mathbb{R})$,

$$\rho := \inf_{v \in S^{d-1-c}} |\mathscr{E}^{(N)}(0)v^N| \Big/ \sup_{x \in B_R} \|\mathscr{E}^{(N+1)}(x)\| \geq \zeta / \sup_{x \in B_R} \|\mathscr{E}^{(N+1)}(x)\|,$$

where we apply the lower bound in (2.4) to obtain the preceding inequality. Therefore,

$$\begin{aligned} |\mathscr{E}(rv)| &\leq \frac{r^N}{N!}|\mathscr{E}^{(N)}(0)v^N| + \frac{r^{N+1}}{N!}\int_0^1 |\mathscr{E}^{N+1}(trv)v^{N+1}|\,dt \quad \text{(by (2.7))} \\ &\leq \frac{r^N}{N!}|\mathscr{E}^{(N)}(0)v^N| + \frac{r^N}{N!}|\mathscr{E}^{(N)}(0)v^N| \quad \text{(by (2.8))}, \end{aligned}$$

that is,

$$\frac{2r^N}{N!}|\mathscr{E}^{(N)}(0)v^N| \geq |\mathscr{E}(rv)|, \quad \forall\, v \in S^{d-1-c} \text{ and } r \in [0, \rho_u). \tag{2.9}$$

[3]For $s, t \in \mathbb{R}$, we denote $s \wedge t = \min\{s, t\}$.

Because $\mathscr{E}^{(n)}(0)v^n = 0$ for $n = 1, \dots, N-1$, the Taylor Formula (2.3) applied to $f(x) = \mathscr{E}'(x)$ with $k = d$ and $M = N-1$ and $x = rv$ yields

$$\mathscr{E}'(rv) = \frac{\mathscr{E}^{(N)}(0)v^{N-1}}{(N-1)!} r^{N-1} + \frac{r^N}{(N-1)!}\int_0^1 \mathscr{E}^{(N+1)}(trv)v^N\,dt, \quad \forall\, v \in S^{d-1-c} \text{ and } r \in [0,R). \tag{2.10}$$

We may choose $\tau \in (0, R \wedge 1]$ small enough that

$$\tau \max_{s\in[0,R]} \|\mathscr{E}^{(N+1)}(sv)v^N\|_{\mathbb{R}^{d*}} \leq \frac{1}{2}\|\mathscr{E}^{(N)}(0)v^{N-1}\|_{\mathbb{R}^{d*}}, \quad \forall\, v \in S^{d-1-c}, \tag{2.11}$$

namely, noting that $\|\mathscr{E}^{(N+1)}(sv)v^N\|_{\mathbb{R}^{d*}} = \sup_{w\in S^d} |\mathscr{E}^{(N+1)}(sv)v^N w|$,

$$\tau := \inf_{v\in S^{d-1-c}} \frac{1}{2}\|\mathscr{E}^{(N)}(0)v^{N-1}\|_{\mathbb{R}^{d*}} \Big/ \sup_{x\in B_R} \|\mathscr{E}^{(N+1)}(x)\| \geq \frac{\zeta}{2} \Big/ \sup_{x\in B_R} \|\mathscr{E}^{(N+1)}(x)\|,$$

where we apply the lower bound in (2.6) to obtain the preceding inequality. Therefore,

$$\begin{aligned}
\|\mathscr{E}'(rv)\|_{\mathbb{R}^{d*}} &\geq \frac{r^{N-1}}{(N-1)!}\|\mathscr{E}^{(N)}(0)v^{N-1}\|_{\mathbb{R}^{d*}} - \frac{r^N}{(N-1)!}\int_0^1 \|\mathscr{E}^{(N+1)}(trv)v^N\|_{\mathbb{R}^{d*}}\,dt \quad \text{(by (2.10))}\\
&\geq \frac{r^{N-1}}{(N-1)!}\|\mathscr{E}^{(N)}(0)v^{N-1}\|_{\mathbb{R}^{d*}} - \frac{\tau_u r^{N-1}}{(N-1)!}\max_{s\in[0,R]}\|\mathscr{E}^{(N+1)}(sv)v^N\|_{\mathbb{R}^{d*}}\\
&\geq \frac{r^{N-1}}{(N-1)!}\|\mathscr{E}^{(N)}(0)v^{N-1}\|_{\mathbb{R}^{d*}} - \frac{r^{N-1}}{2(N-1)!}\|\mathscr{E}^{(N)}(0)v^{N-1}\|_{\mathbb{R}^{d*}} \quad \text{(by (2.11))}\\
&= \frac{r^{N-1}}{2(N-1)!}\|\mathscr{E}^{(N)}(0)v^{N-1}\|_{\mathbb{R}^{d*}},
\end{aligned}$$

for all $v \in S^{d-1-c}$ and $r \in [0,\tau)$, and thus,

$$\|\mathscr{E}'(rv)\|_{\mathbb{R}^{d*}} \geq \frac{r^{N-1}}{2N!}\|\mathscr{E}^{(N)}(0)v^{N-1}\|_{\mathbb{R}^{d*}}, \quad \forall\, v \in S^{d-1-c} \text{ and } r \in [0,\tau). \tag{2.12}$$

Define

$$\sigma := \rho \wedge \tau \in (0, R \wedge 1]. \tag{2.13}$$

We compute that, for all $v \in S^{d-1-c}$ and $r \in [0, \sigma)$,

$$\begin{aligned}
\|\mathscr{E}'(rv)\|_{\mathbb{R}^{d*}} &\geq \frac{r^{N-1}}{2N!}\|\mathscr{E}^{(N)}(0)v^{N-1}\|_{\mathbb{R}^{d*}} \quad \text{(by (2.12))} \\
&= \frac{1}{2N!}\|\mathscr{E}^{(N)}(0)v^{N-1}\|_{\mathbb{R}^{d*}} \left(\frac{2}{N!}\|\mathscr{E}^{(N)}(0)v^{N-1}\|_{\mathbb{R}^{d*}}\right)^{-(N-1)/N} \\
&\quad \times \left(\frac{2r^N}{N!}\|\mathscr{E}^{(N)}(0)v^{N-1}\|_{\mathbb{R}^{d*}}\right)^{(N-1)/N} \\
&= \frac{2^{1/N}}{4}\left(\frac{1}{N!}\|\mathscr{E}^{(N)}(0)v^{N-1}\|_{\mathbb{R}^{d*}}\right)^{1/N}\left(\frac{2r^N}{N!}\|\mathscr{E}^{(N)}(0)v^{N-1}\|_{\mathbb{R}^{d*}}\right)^{(N-1)/N} \\
&\geq \frac{2^{1/N}}{4}\left(\frac{1}{N!}\|\mathscr{E}^{(N)}(0)v^{N-1}\|_{\mathbb{R}^{d*}}\right)^{1/N}\left(\frac{2r^N}{N!}|\mathscr{E}^{(N)}(0)v^{N}|\right)^{(N-1)/N} \quad \text{(by (2.5))} \\
&\geq \frac{2^{1/N}}{4}\left(\frac{1}{N!}\|\mathscr{E}^{(N)}(0)v^{N-1}\|_{\mathbb{R}^{d*}}\right)^{1/N} |\mathscr{E}(rv)|^{(N-1)/N} \quad \text{(by (2.9))}.
\end{aligned}$$

This yields (2.2) with $\theta = (N-1)/N \in [1/2, 1)$, for all $v \in S^{d-1-c}$, and

$$C := \inf_{v \in S^{d-1-c}} \frac{2^{1/N}}{4}\left(\frac{1}{N!}\|\mathscr{E}^{(N)}(0)v^{N-1}\|_{\mathbb{R}^{d*}}\right)^{1/N} \geq \frac{2^{1/N}}{4}\left(\frac{\zeta}{N!}\right)^{1/N}. \tag{2.14}$$

By the reductions described earlier, this completes the proof of Theorem 2.5. □

3. Łojasiewicz gradient inequality for C^1 functions with simple normal crossings

In this section, we prove Theorem 3 using a simple, coordinate-based alternative to an argument due to Bierstone and Milman of their more general [9, Theorem 2.7].

Proof of Theorem 3. By hypothesis, the function $\mathscr{E} : U \to \mathbb{R}$ has simple normal crossings in the sense of Definition 1.1 and $\mathscr{E}(0) = 0$. Therefore,

$$\mathscr{E}(x) = \mathscr{F}(x)\prod_{i=1}^{c} x_i^{n_i}, \quad \forall\, x \in U, \tag{3.1}$$

for integers $c \geq 1$ with $c \leq d$ and $n_i \geq 1$ and a C^1 function $\mathscr{F} : U \to \mathbb{R}$ with $\mathscr{F}(x) \neq 0$ for all $x \in U$.[4] Hence, if $\{e_i\}_{i=1}^d$ and $\{e_i^*\}_{i=1}^d$ denote the standard basis and dual basis, respectively, for $\mathbb{R}^d$ and $\mathbb{R}^{d*}$, then the differential of $\mathscr{E}$ is given by

$$\begin{aligned}
\mathscr{E}'(x) &= \sum_{j=1}^{d} \mathscr{E}_{x_j}(x) e_j^* \\
&= \sum_{j=1}^{c}(x_j^{n_j}\mathscr{F}_{x_j}(x) + n_j x_j^{n_j-1}\mathscr{F}(x)) \prod_{\substack{i=1 \\ i \neq j}}^{c} x_i^{n_i}\, e_j^* + \prod_{i=1}^{c} x_i^{n_i} \sum_{j=c+1}^{d} \mathscr{F}_{x_j}(x) e_j^*,
\end{aligned}$$

[4] By making a further coordinate change, one could assume that $\mathscr{F} = 1$ without loss of generality but we shall omit that step.

that is,

$$\mathscr{E}'(x) = \prod_{i=1}^{c} x_i^{n_i} \sum_{j=1}^{c} (x_j \mathscr{F}_{x_j}(x) + n_j \mathscr{F}(x)) x_j^{-1} e_j^* + \prod_{i=1}^{c} x_i^{n_i} \sum_{j=c+1}^{d} \mathscr{F}_{x_j}(x) e_j^*, \quad \forall\, x \in U, \tag{3.2}$$

where the sum over $j = c+1, \ldots, d$ are omitted if $c = d$. Observe that

$$\|\mathscr{E}'(x)\|_{\mathbb{R}^{d*}}^2 \geq \prod_{i=1}^{c} x_i^{2n_i} \sum_{j=1}^{c} (x_j \mathscr{F}_{x_j}(x) + n_j \mathscr{F}(x))^2 x_j^{-2}, \quad \forall\, x \in U. \tag{3.3}$$

Because $\mathscr{F}(0) \neq 0$ and $\mathscr{F}$ is C^1, there is a constant $\sigma \in (0,1]$ such that $B_\sigma \Subset U$ and

$$|x_j \mathscr{F}_{x_j}(x)| \leq \frac{n_j}{2} |\mathscr{F}(x)|, \quad \forall\, x \in B_\sigma \text{ and } j = 1, \ldots, c,$$

and thus

$$|x_j \mathscr{F}_{x_j}(x) + n_j \mathscr{F}(x)| \geq \frac{n_j}{2} |\mathscr{F}(x)|, \quad \forall\, x \in B_\sigma \text{ and } j = 1, \ldots, c.$$

Hence (3.3), noting that $n_j \geq 1$ for $j = 1, \ldots, c$, yields the lower bound

$$\|\mathscr{E}'(x)\|_{\mathbb{R}^{d*}}^2 \geq \frac{\mathscr{F}(x)^2}{4} \prod_{i=1}^{c} x_i^{2n_i} \sum_{j=1}^{c} x_j^{-2}, \quad \forall\, x \in B_\sigma. \tag{3.4}$$

On the other hand, (3.1) gives

$$\mathscr{E}(x)^2 = \mathscr{F}(x)^2 \prod_{i=1}^{c} x_i^{2n_i}, \quad \forall\, x \in U. \tag{3.5}$$

Define

$$m := \inf_{x \in B_\sigma} |\mathscr{F}(x)| > 0 \quad \text{and} \quad M := \sup_{x \in B_\sigma} |\mathscr{F}(x)| < \infty. \tag{3.6}$$

Because $\mathscr{E}'(0) = 0$, we must have $c \geq 2$ or $c = 1$ and $n_1 \geq 2$ by examining the expression (3.2) for $\mathscr{E}'(x)$ when $x = 0$. If $c = 1$, then $n_1 \geq 2$, then inequalities (3.4), (3.5), and (3.6) give

$$\|\mathscr{E}'(x)\|_{\mathbb{R}^{d*}} \geq \frac{1}{2m} |x_1|^{n_1 - 1} \quad \text{and} \quad |\mathscr{E}(x)| \leq M |x_1|^{n_1}, \quad \forall\, x \in B_\sigma.$$

Combining these inequalities yields

$$\|\mathscr{E}'(x)\|_{\mathbb{R}^{d*}} \geq \frac{m}{2M^{(n_1-1)/n_1}} |\mathscr{E}(x)|^{(n_1-1)/n_1}, \quad \forall\, x \in B_\sigma,$$

and hence we obtain (1.5) with $\theta = 1 - 1/n_1$ and $C_0 = m/(2M^\theta)$ if $c = 1$.

For the remainder of the proof, we assume $c \geq 2$ and recall the *Generalized Young Inequality*,

$$\left(\prod_{j=1}^{c} a_j \right)^r \leq r \sum_{j=1}^{c} \frac{a_j^{p_j}}{p_j}, \tag{3.7}$$

for constants $a_j > 0$ and $p_j > 0$ and $r > 0$ such that $\sum_{j=1}^{c} 1/p_j = 1/r$. (See Remark 3.1.) For

$$N := \sum_{j=1}^{c} n_j,$$

we observe that the inequality,

$$\prod_{j=1}^{c} x^{-2n_j/N} \leq \frac{1}{N} \sum_{j=1}^{c} n_j x_j^{-2}, \quad \text{for } x_j \neq 0 \text{ with } j = 1, \ldots, c, \tag{3.8}$$

follows from (3.7) by substituting $r = 1$ and $a_j = x_j^{-2n_j/N}$ (with $x_j \neq 0$) and $p_j = N/n_j$ for $j = 1, \dots, c$ in (3.7). Setting

$$n := \max_{1\le j\le c} n_j \quad \text{and} \quad \theta := 1 - 1/N \in [1/2, 1)$$

and applying (3.8) yields

$$\prod_{i=1}^c x_i^{2n_i} \sum_{j=1}^c x_j^{-2} \geq \frac{N}{n} \prod_{i=1}^c x_i^{2n_i(1-1/N)},$$

that is,

$$\prod_{i=1}^c x_i^{2n_i} \sum_{j=1}^c x_j^{-2} \geq \frac{N}{n} \left(\prod_{i=1}^c x_i^{2n_i}\right)^\theta, \quad \forall\, x \in \mathbb{R}^c. \tag{3.9}$$

We now combine inequalities (3.4), (3.5), (3.6), and (3.9) to give

$$\|\mathscr{E}'(x)\|^2_{\mathbb{R}^{d*}} \geq \frac{m^2 N}{4n} \left(\prod_{i=1}^c x_i^{2n_i}\right)^\theta \quad \text{and} \quad \mathscr{E}(x)^{2\theta} \leq M^{2\theta} \left(\prod_{i=1}^c x_i^{2n_i}\right)^\theta, \quad \forall\, x \in B_\sigma.$$

Taking square roots and combining the preceding two inequalities yields (1.5) with constant $C_0 = m\sqrt{N/n}/(2M^\theta)$ if $c \geq 2$. This completes the proof of Theorem 3. □

Remark 3.1 (Generalized Young Inequality). The inequality (3.7) may be deduced from Hardy, Littlewood, and Pólya [54, Inequality (2.5.2)],

$$\prod_{i=1}^c b_i^{q_i} \leq \sum_{i=1}^c q_i b_i, \tag{3.10}$$

where $b_i > 0$ and $c \geq 1$ and $q_i > 0$ and $\sum_{i=1}^c q_i = 1$. Indeed, set $a_i = b_i^{q_i/r}$, so $b_i = a_i^{r/q_i}$, and $p_i = r/q_i$ to give

$$\prod_{i=1}^c a_i^r \leq \sum_{i=1}^c q_i a_i^{p_i}.$$

But $q_i = r/p_i$ and thus

$$\left(\prod_{i=1}^c a_i\right)^r \leq r \sum_{i=1}^c \frac{1}{p_i} a_i^{p_i},$$

which is (3.7); see also [54, Section 8.3]. The inequality (3.7) is proved directly by Haraux as [46, Lemma 3.2] by using concavity of the logarithm function on $(0, \infty)$.

4. Resolution of singularities and application to the Łojasiewicz gradient inequality

We begin in Sections 4.1 and 4.2 by recalling the definitions of divisors and ideals, respectively, with simple normal crossings. In Section 4.3, we recall a statement of resolution of singularities for analytic varieties and in Section 4.4, we apply that to prove Theorem 1 as a corollary of Theorem 3. Unless stated otherwise, 'analytic' may refer to real or complex analytic in this section.

4.1. **Divisors with simple normal crossings.** For basic methods of and notions in algebraic geometry — including blowing up, divisors, and morphisms — we refer to Griffiths and Harris [43], Hartshorne [55], and Shafarevich [106, 107]. For terminology regarding real analytic varieties, we refer to Guaraldo, Macrì, and Tancredi [44]; see also Griffiths and Harris [43] and Grauert and Remmert [40] for complex analytic varieties.

Following Griffiths and Harris [43, pp. 12–14, pp. 20–22, and pp. 129–130] (who consider *complex* analytic subvarieties of smooth *complex* manifolds), let M be a (real or complex) analytic (not necessarily compact) manifold of dimension $d \geq 1$ and $V \subset M$ be an *analytic subvariety*, that is, for each point $p \in V$, there is an open neighborhood $U \subset M$ of p and a finite collection, $\{f_1, \dots, f_k\}$ (where k may depend on p), of analytic functions on U such that $V \cap U = f_1^{-1}(0) \cap \cdots \cap f_k^{-1}(0)$. One calls p a *smooth point of* V if $V \cap U$ is cut out transversely by $\{f_1, \dots, f_k\}$, that is, if the $k \times d$ matrix $(\partial f_i/\partial x_j)(p)$ has rank k, in which case (possibly after shrinking U), we have that $V \cap U$ is an analytic (smooth) submanifold of codimension k in U. An analytic subvariety $V \subset M$ is called *irreducible* if V cannot be written as the union of two analytic subvarieties, $V_1, V_2 \subset M$, with $V_i \neq V$ for $i = 1, 2$.

One calls $V \subset M$ an *analytic subvariety of dimension* $d-1$ if V is a *analytic hypersurface*, that is, for any point $p \in V$, then $U \cap V = f^{-1}(0)$, for some open neighborhood, $U \subset M$ of p, and some analytic function, f, on U [43, p. 20]. We then recall the

Definition 4.1 (Divisor on an analytic manifold)**.** (See [43, p. 130].) A *divisor* D on an analytic manifold M is a locally finite, formal linear combination,

$$D = \sum_i a_i V_i,$$

of irreducible, analytic hypersurfaces of M, where $a_i \in \mathbb{Z}$.

We can now state the

Definition 4.2 (Simple normal crossing divisor)**.** (See Kollár [70, Definition 3.24].) Let X be a smooth algebraic variety of dimension $d \geq 1$. One says that $E = \sum_i E_i$ is a *simple normal crossing divisor* on X if each E_i is smooth, and for each point $p \in X$ one can choose local coordinates, $x_1, \dots, x_d$ in the maximal ideal $\mathfrak{m}_p$ of the local ring, $\mathscr{O}_p$, of regular functions defined on some open neighborhood U of $p \in X$ such that for each i the following hold:

(1) Either $p \notin E_i$, or
(2) $E_i \cap U = \{q \in U : x_{j_i}(q) = 0\}$ in an open neighborhood $U \subset X$ of p for some j_i, and
(3) $j_i \neq j_{i'}$ if $i \neq i'$.

A subvariety, $Z \subset X$, has *simple normal crossings* with E if one can choose $x_1, \dots, x_d$ as above such that in addition

(4) $Z = \{q \in U : x_{j_1}(q) = \cdots x_{j_s}(q) = 0\}$ for some $j_1, \dots, j_s$.

In particular, Z is smooth, and some of the E_i are allowed to contain Z.

Kollár also gives the following, more elementary definition that serves, in part, to help compare the concepts of simple normal crossing divisor (as used by [69, 117]) and normal crossing divisor (as used by [9]), in the context of resolution of singularities.

Definition 4.3 (Simple normal crossing divisor)**.** (See Kollár [70, Definition 1.44].) Let X be a smooth algebraic variety of dimension $d \geq 1$ and $E \subset X$ a divisor. One calls E a *simple normal crossing divisor* if every irreducible component of E is smooth and all intersections are transverse. That is, for every point $p \in E$ we can choose local coordinates, $x_1, \dots, x_d$ on an open neighborhood $U \subset X$ of p, and $m_i \in \mathbb{Z} \cap [0, \infty)$ for $i = 1, \dots, d$ such that $U \cap E = \{q \in U : \prod_{i=1}^d x^{m_i}(q) = 0\}$.

Remark 4.4 (Normal crossing divisor). (See Kollár [70, Remark 1.45].) Continuing the notation of Definition 4.3, one calls E a *normal crossing divisor* if for every $p \in E$ there are local *analytic* or formal coordinates, $x_1, \dots, x_d$, and natural numbers $m_1, \dots, m_d$ such that $U \cap E = \{q \in U : \prod_{i=1}^d x^{m_i}(q) = 0\}$.

Definitions 4.2 and 4.3 extend to the categories of analytic varieties, where $\mathscr{O}_p$ is then the local ring of analytic functions; see, for example, Kollár [70, Section 3.44]. In the category of analytic varieties, Remark 4.4 implies that the concepts of simple normal crossing divisor and normal crossing divisor coincide.[5] Definitions of simple normal crossing divisors are also provided by Cutkowsky [27, Exercise 3.13 (2)], Hartshorne [55, Remark 3.8.1] and Lazarsfeld [77, Definition 4.1.1].

4.2. **Ideals with simple normal crossings.** For our application to the proof of the gradient inequality, we shall need to more generally consider ideals with simple normal crossings and the corresponding statement of resolution of singularities. We review the concepts that we shall require for this purpose. For the theory of ringed spaces, sheaf theory, analytic spaces, and analytic manifolds we refer to Grauert and Remmert [41], Griffiths and Harris [43], and Narasimhan [92] in the complex analytic category and Guaraldo, Macrì, and Tancredi [44] in the real analytic category; see also Hironaka et al. [3, 4, 61]. If X is an analytic manifold, then $\mathscr{O}_X$ is the sheaf of analytic functions on X. An ideal $\mathscr{I} \subset \mathscr{O}_X$ is *locally finite* if for every point $p \in X$, there are an open neighborhood, $U \subset X$, and a finite set of analytic functions, $\{f_1, \dots, f_k\} \subset \mathscr{O}_U$, such that

$$\mathscr{I} = f_1\mathscr{O}_U + \cdots + f_k\mathscr{O}_U,$$

and *locally principal* if $k = 1$ for each point $p \in X$.

If $p \in X$, then $\mathscr{O}_p$, is the ring of (germs of) analytic functions defined on some open neighborhood of p. The quotient sheaf, $\mathscr{O}_X/\mathscr{I}$ is a sheaf of rings on X and its *support*,

$$Z := \operatorname{supp}(\mathscr{O}_X/\mathscr{I}),$$

is the set of all points $p \in X$ where $(\mathscr{O}_X/\mathscr{I})_p \neq 0$, that is, where $\mathscr{I}_p \neq \mathscr{O}_p$. In an open neighborhood U of p one has

$$Z \cap U = f_1^{-1}(0) \cap \cdots \cap f_k^{-1}(0),$$

so locally Z is the zero set of finitely many analytic functions.

In order to state the version of resolution of singularities that we shall need, we recall some definitions from Cutkosky [27, pp. 40–41] and Kollár [70, Note on Terminology 3.16], given here in the real or complex analytic category, rather than the algebraic category, for consistency with our application. Suppose that X is a non-singular variety and $\mathscr{I} \subset \mathscr{O}_X$ is an ideal sheaf; a *principalization of the ideal* $\mathscr{I}$ is a proper birational morphism, $\pi : \widetilde{X} \to X$, such that $\widetilde{X}$ is non-singular and

$$\pi^*\mathscr{I} \subset \mathscr{O}_{\widetilde{X}}$$

is a locally principal ideal. If X is a non-singular variety of dimension d and $\mathscr{I} \subset \mathscr{O}_X$ is a locally principal ideal, then one says that $\mathscr{I}$ has *simple normal crossings* (or *monomial*) *at a point* $p \in X$ if there exist local coordinates, $\{x_1, \dots, x_d\} \subset \mathscr{O}_p$, such that

$$\mathscr{I}_p = x_1^{m_1} \cdots x_n^{m_d}\mathscr{O}_p,$$

for some $m_i \in \mathbb{Z} \cap [0, \infty)$ with $i = 1, \dots, d$. One says that $\mathscr{I}$ is *locally monomial* if it is monomial at every point $p \in X$ or, equivalently, if it is the ideal sheaf of a *simple normal crossing divisor* in the sense of Definition 4.2.

[5] I am grateful to Jarosław Włodarczyk for clarifying this point.

Suppose that D is an effective divisor on a non-singular variety X of dimension n, so $D = m_1E_1 + \cdots + m_dE_d$, where E_i are irreducible, codimension-one subvarieties of X, and $m_i \in \mathbb{Z} \cap [0, \infty)$ with $i = 1, \ldots, d$. One says that D has *simple normal crossings* if

$$\mathscr{I}_D = \mathscr{I}_{E_1}^{m_1} \cdots \mathscr{I}_{E_d}^{m_d}$$

has *simple normal crossings*.

4.3. **Resolution of singularities.** We recall from Cutkosky [27, pp. 40–41] that a *resolution of singularities* of an algebraic or analytic variety X is a proper birational morphism, $\pi : \widetilde{X} \to X$, such that $\widetilde{X}$ is non-singular. Hironaka [58] proved that any algebraic variety over any field of characteristic zero admits a resolution of singularities and, moreover, that both complex and real analytic varieties admit resolutions of singularities as well [3, 4, 61]. Bierstone and Milman [9] (see [10] for their expository introduction to [9]) have developed a proof of resolution of singularities that applies to real and complex analytic varieties and to algebraic varieties over any field of characteristic zero and which significantly shortens and simplifies Hironaka's proof. Additional references for resolution of singularities include Cutkowsky [27], Faber and Hauser [31], Hauser [56], Hironaka [57], Kollár [70], Villamayor [114, 115, 30], and Włodarczyk [116, 117]. Proofs of special cases of resolution of singularities for real analytic varieties were previously provided by Bierstone and Milman [7, 8]. The most useful version of resolution of singularities for our application is

Theorem 4.5 (Principalization and monomialization of an ideal sheaf)**.** *(See Bierstone and Milman* [9, Theorem 1.10]*, Kollár* [70, Theorems 3.21 and 3.26 and p. 135 and Section 3.44] *and Włodarczyk* [117, Theorem 2.0.2] *for analytic varieties; compare Włodarczyk* [116, Theorem 1.0.1] *for algebraic varieties.) If X is a smooth analytic variety and $\mathscr{I} \subset \mathscr{O}_X$ is a nonzero ideal sheaf, then there are a smooth analytic variety, $\widetilde{X}$, and a birational and projective morphism, $\pi : \widetilde{X} \to X$, such that*

(1) $\pi^*\mathscr{I} \subset \mathscr{O}_{\widetilde{X}}$ is the ideal sheaf of a simple normal crossing divisor,

(2) $\pi : \widetilde{X} \to X$ is an isomorphism over $X \setminus \operatorname{cosupp} \mathscr{I}$, where $\operatorname{cosupp} \mathscr{I}$ (or $\operatorname{supp}(\mathscr{O}_X/\mathscr{I})$) is the cosupport of $\mathscr{I}$.

Versions of Theorem 4.5 when X is an algebraic surface over a field of characteristic zero are provided by Cutkosky [27, p. 29] and Kollár [70, Theorem 1.74]. Kashiwara and Schapira [67] provide the following useful variant of Theorem 4.5.

Proposition 4.6 (Desingularization for the zero set of a real analytic function and its gradient map)**.** *(See Kashiwara and Schapira* [67, Proposition 8.2.4]*.) Let X be a real analytic manifold and $f : X \to \mathbb{R}$ be a real analytic function that is not identically zero on each connected component of X. Set $Z = \{x \in X : f(x) = 0$ and $df(x) = 0\}$. Then there exists a proper morphism of real analytic manifolds, $\pi : Y \to X$, which induces a real analytic diffeomorphism, $Y \setminus \pi^{-1}(Z) \cong X \setminus Z$, such that in an open neighborhood of each point $y_0 \in \pi^{-1}(Z)$ there exists a local coordinate system, $(y_1, \ldots, y_d)$, with $f \circ \pi(y) = y_1^{n_1} \cdots y_d^{n_d}$, for some $n_i \in \mathbb{Z} \cap [0, \infty)$ with $i = 1, \ldots, d$.*

4.4. **Application to the Łojasiewicz gradient inequality.** We can now conclude the

Proof of Theorem 1. As in the proof of Theorem 2.2, we may assume without loss of generality that $\mathscr{E}(0) = 0 \in \mathbb{R}$ and $\mathscr{E}'(0) = 0 \in \mathbb{R}^{d*}$. Define $\mathscr{I} := \mathscr{E}\mathscr{O}_U$ to be the ideal in $\mathscr{O}_U$ generated by $\mathscr{E}$, with support of $\mathscr{O}_U/\mathscr{I}$ given by $Z = \mathscr{E}^{-1}(0)$. Let $\pi : \widetilde{U} \to U$ be a resolution of singularities provided by Theorem 4.5, so

$$\pi^*\mathscr{I} = \tilde{\mathscr{E}}\mathscr{O}_{\widetilde{U}}$$

is the ideal sheaf of a simple normal crossing divisor, where $\tilde{\mathscr{E}} := \mathscr{E} \circ \pi$ and

$$\pi : \widetilde{U} \setminus E \cong U \setminus Z,$$

is a real analytic diffeomorphism, with

$$E := \pi^{-1}(Z) = \{\tilde{x} \in \widetilde{U} : \tilde{\mathscr{E}}(\tilde{x}) = 0\} \subset \widetilde{U}$$

denoting the exceptional divisor (with ideal $\pi^*\mathscr{I}$).

By assumption, $0 \in Z$ and we may further assume without loss of generality that $0 \in \pi^{-1}(0) \subset E$ and $\widetilde{U} \subset \mathbb{R}^d$ is an open neighborhood of the origin, possibly after shrinking U and hence $\widetilde{U}$. By Theorem 4.5, the function $\tilde{\mathscr{E}}$ is the product of a monomial in the coordinate functions, $x_1, \ldots, x_d$, and a real analytic function $\mathscr{F}$ that is non-zero at the origin. In particular, $\tilde{\mathscr{E}}$ has simple normal crossings in the sense of Definition 1.1, possibly after further shrinking U and hence $\widetilde{U}$, so $\mathscr{F}(\tilde{x}) \neq 0$ for all $\tilde{x} \in \widetilde{U}$. We can thus apply Theorem 3 to $\tilde{\mathscr{E}} = \mathscr{E} \circ \pi$ and obtain

$$\|(\mathscr{E} \circ \pi)'(\tilde{x})\|_{\mathbb{R}^{d*}} \geq C|(\mathscr{E} \circ \pi)(\tilde{x})|^\theta, \quad \forall\, \tilde{x} \in B_\delta,$$

for constants $C \in (0, \infty)$ and $\theta \in [1/2, 1)$ and $\delta \in (0, 1]$. Now $(\mathscr{E} \circ \pi)(\tilde{x}) = \mathscr{E}(x)$ for $x \in U$ and $\tilde{x} \in \pi^{-1}(x)$ and therefore the preceding gradient inequality yields

$$\|(\mathscr{E} \circ \pi)'(\tilde{x})\|_{\mathbb{R}^{d*}} \geq C|\mathscr{E}(x)|^\theta, \quad \forall\, x \in B_\sigma \text{ and } \tilde{x} \in \pi^{-1}(x), \tag{4.1}$$

where $\sigma \in (0, 1]$ is chosen small enough that $B_\delta \supset \pi^{-1}(B_\sigma)$. The Chain Rule gives

$$\begin{aligned}\|(\mathscr{E} \circ \pi)'(\tilde{x})\|_{\mathbb{R}^{d*}} &\leq \|\mathscr{E}'(\pi(\tilde{x}))\|_{\mathbb{R}^{d*}} \|\pi'(\tilde{x})\|_{\mathrm{End}(\mathbb{R}^d)} \\ &\leq M\|\mathscr{E}'(\pi(\tilde{x}))\|_{\mathbb{R}^{d*}} \quad \forall\, \tilde{x} \in \widetilde{U},\end{aligned}$$

where $M := \sup_{\tilde{x} \in B_\delta} \|\pi'(\tilde{x})\|_{\mathrm{End}(\mathbb{R}^d)}$. Because $\pi(\tilde{x}) = x \in U$, the preceding inequality simplifies:

$$\|(\mathscr{E} \circ \pi)'(\tilde{x})\|_{\mathbb{R}^{d*}} \leq M\|\mathscr{E}'(x)\|_{\mathbb{R}^{d*}}, \quad \forall\, \tilde{x} \in \widetilde{U}. \tag{4.2}$$

By combining the inequalities (4.1) and (4.2), we obtain

$$\|\mathscr{E}'(x)\|_{\mathbb{R}^{d*}} \geq (C/M)|\mathscr{E}(x)|^\theta, \quad \forall\, x \in B_\sigma,$$

which is (1.1), as desired. □

We can also complete the

Proof of Corollary 2. The conclusions follow by combining the proof of Theorem 1 (which shows that the Łojasiewicz exponent is preserved by the resolution morphism) and Theorem 3. □

5. Łojasiewicz distance inequalities

It remains to prove the distance inequalities (Corollaries 4 and 5). For this purpose, the proof of [9, Theorem 2.8] (see also [89]) applies but we shall include additional details for completeness. We assume a Łojasiewicz exponent[6] $\theta \in [1/2, 1)$, denoted by $\mu = 1 - \theta \in (0, 1/2]$ in [9].

Proof of Corollary 4. As usual, we may assume without loss of generality that $\mathscr{E}(0) = 0 \in \mathbb{R}$. Consider first the case where, in addition, $\mathscr{E}'(0) = 0 \in \mathbb{R}^{d*}$. Note that

$$B_\sigma \cap \operatorname{Crit} \mathscr{E} = \{x \in B_\sigma : \mathscr{E}'(x) = 0\} \subset B_\sigma \cap \mathscr{E}^{-1}(0)$$

[6]We exclude the trivial case $\theta = 1$ and $\mathscr{E}'(0) \neq 0$.

by the gradient inequality (1.1) with exponent $\theta \in [1/2, 1)$. Consider a point $x_0 \in B_\sigma \subset \mathbb{R}^d$ such that $\mathscr{E}(x_0) \neq 0$ and thus $\mathscr{E}'(x_0) \neq 0$ by (1.1). We may assume without loss of generality that $\mathscr{E}(x_0) > 0$ (otherwise, replace $\mathscr{E}$ by $-\mathscr{E}$). Let $\{x(t) : 0 \leq t < T\}$ be a solution to

$$\frac{dx}{dt} = -\frac{\mathscr{E}'(x(t))}{\|\mathscr{E}'(x(t))\|_{\mathbb{R}^{d*}}}, \quad x(0) = x_0,$$

where $T \in (0, \infty]$ is the smallest time such that $\mathscr{E}'(x(T)) = 0$. Write $Q(t) = \mathscr{E}(x(t))$ and observe that

$$Q'(t) = \mathscr{E}'(x(t))x'(t) = -\|\mathscr{E}'(x(t))\|_{\mathbb{R}^{d*}} < 0.$$

Now $Q(0) = \mathscr{E}(x_0) > 0$ and $0 < Q(t) \leq Q(0)$ for all $t \in [0, t_0)$, where $t_0 \in (0, T]$ is the smallest time such that $Q(t_0) = 0$ (and thus $x(t_0) \in \mathscr{E}^{-1}(0)$). But we have

$$\begin{aligned}
\frac{\mathscr{E}(x_0)^{1-\theta}}{1-\theta} &\geq \frac{Q(0)^{1-\theta} - Q(t)^{1-\theta}}{1-\theta} \\
&= -\frac{1}{1-\theta}\int_0^t \frac{d}{ds}Q(s)^{1-\theta}\,ds = -\int_0^t Q(s)^{-\theta}Q'(s)\,ds \\
&= \int_0^t \mathscr{E}(x(s))^{-\theta}\|\mathscr{E}'(x(s))\|_{\mathbb{R}^{d*}}\,ds \\
&\geq \int_0^t C_0\,ds = C_0\,t, \quad 0 \leq t < t_0 \quad \text{(by Theorem 1)}.
\end{aligned}$$

It follows that $t_0 < \infty$ and so the solution curve $x(t)$ tends to a point $x(t_0) \in \mathscr{E}^{-1}(0) \cap B_\sigma$ in a finite time t_0. Since $|x'(t)| = 1$, then $x(t)$ is parameterized by arc-length and

$$t_0 = \text{Length}(x(t) : 0 \leq t \leq t_0) \geq \text{dist}(x_0, \mathscr{E}^{-1}(0)).$$

We thus obtain

$$\mathscr{E}(x_0)^{1-\theta} \geq (1-\theta)C_0\ \text{dist}(x_0, \mathscr{E}^{-1}(0)),$$

and this is (1.6), as desired, with exponent $\alpha = 1/(1-\theta) \in [2, \infty)$.

If $\mathscr{E}(0) = 0$ but $\mathscr{E}'(0) \neq 0$, we may consider $\mathscr{F}(x) := \mathscr{E}(x)^2$ for $x \in U$ and observe that $\mathscr{F}'(0) = 0$ and $\mathscr{F}(0) = 0$ and $\mathscr{E}^{-1}(0) \cap B_\sigma = \mathscr{F}^{-1}(0) \cap B_\sigma$, so the preceding argument applies to $\mathscr{F}$ with $\theta \in [1/2, 1)$ determined by $\mathscr{F}$ to give

$$\mathscr{E}(x_0)^{2(1-\theta)} \geq (1-\theta)C_0\ \text{dist}(x_0, \mathscr{E}^{-1}(0)),$$

and this is (1.6), as desired, with exponent $\alpha = 1/(2(1-\theta)) \in [1, \infty)$. This completes the proof of Corollary 4. □

Lastly, we give the

Proof of Corollary 5. As usual, we may assume without loss of generality that $\mathscr{E}(0) = 0 \in \mathbb{R}$ and $\mathscr{E}'(0) = 0 \in \mathbb{R}^{d*}$ and note that $B_{\sigma_2} \cap \operatorname{Crit}\mathscr{E} \subset B_{\sigma_2} \cap \mathscr{E}^{-1}(0)$, for small enough $\sigma_2 \in (0, 1]$, by Theorem 1. We then combine the Łojasiewicz gradient and distance inequalities, (1.1) and (1.6), to give, for a possibly smaller $\sigma_2 \in (0, 1]$,

$$\begin{aligned}
\|\mathscr{E}'(x)\|_{\mathbb{R}^{d*}} &\geq C_0|\mathscr{E}(x)|^\theta \\
&\geq C_0\left(C_1\,\text{dist}(x, \operatorname{Crit}\mathscr{E})^\alpha\right)^\theta, \quad \forall\, x \in B_{\sigma_2}.
\end{aligned}$$

Since $\alpha \in [2, \infty)$ by the proof of Corollary 4 (because $\mathscr{E}'(0) = 0$ in addition to $\mathscr{E}(0) = 0$) and $\theta \in [1/2, 1)$, then $\beta = \alpha\theta \in [1, \infty)$. □

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